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A Group with Trivial Centre Satisfying the Normalizer Condition

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This note is devoted to the construction of a group G with the following properties:

- (I) G' is of exponent p and abelian
- (II) Every proper subgroup of G is subnormal and nilpotent
- (III) $Z(G) = 1$.

The existence of such a group has the following consequences:

- (a) A group satisfying the normalizer condition need not be hypercentral (see problem 21 in the Kurosh-Černikov survey [3], see also Kurosh [1], p. 226 and [2], p. 392).
- (b) A group with all its proper subgroups nilpotent and subnormal need not be nilpotent (see problem 22 in [3]) nor need it be hypercentral.
- (c) The direct product of two groups satisfying the normalizer condition need not satisfy the normalizer condition (this is part of problem 17 in [3]). In fact, the direct square of the group G does not satisfy the normalizer condition because $Z(G) = 1$.
- (d) If all subgroups of a group are subnormal, there need not be a bound on the defects of these subgroups (cf. Roseblade [4], p. 402).

First we will provide some preliminary statements which explain why groups satisfying condition (I) may be considered as test cases for soluble groups

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satisfying condition (II) or which will prove helpful in the proof of the properties of the group to be constructed. The group itself is given in two presentations, and its properties are proved in whichever presentation it is more convenient.

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If x, y are two elements, we will denote their commutator by

$$x \circ y = x^{(1)} \circ y = x^{-1}y^{-1}xy;$$

and we define inductively $x^{(n)} \circ y = x \circ (x^{(n-1)} \circ y)$.

We begin with a statement exhibiting the existence of certain metabelian quotient groups in non-nilpotent soluble groups satisfying condition (II).

LEMMA 1. *If G is a non-nilpotent soluble group satisfying condition (II) and $G' \neq G$, then*

- (a) *If $G = \{U, V\}$ then $U = G$ or $V = G$.*
- (b) *G/G' is isomorphic to C_{p^∞} for some prime p .*
- (c) *G is a countable p -group.*
- (d) *$(G')^p \neq G'$.*

Proof. For (a), assume first the existence of $U \neq G$ and $V \neq G$ such that $G = \{U, V\}$. By condition (II), U and V are contained in proper normal subgroups U^* and V^* of G , which are nilpotent by condition (II). But then by Fitting's Theorem $G = U^*V^*$ is nilpotent contrary to hypothesis. So (a) is true.

It follows from (a) that G/G' is not factorizable, and so (see Scott [5], (13.1.6), p. 374) G/G' is either cyclic of prime power order or isomorphic to C_{p^∞} for some prime p . If G/G' is cyclic, there is an element x in G such that $G = \{x, G'\}$. But $G' \neq G$ and $\{x\} \neq G$ (for G is not abelian), contradicting (a), and we are left with statement (b).

It is a consequence of (a) that any set of representatives of the cosets of G' generate the whole group G . Therefore G can be generated by countably many elements and is itself countable.

Consider now $H = G/G''$ and choose an element x of H . By condition (II) the subgroup $\{x\}$ of H is subnormal, therefore there is an integer n (depending on x) such that

$$(*) \quad x^{(n)} \circ y = 1 \text{ for all } y \text{ in } H.$$

Furthermore there is a power m of p such that x^m is contained in H' . If y is an element of H' and $n > 1$, then

$$1 = (x^m) \circ (x^{(n-2)} \circ y) = (x^{(n-1)} \circ y)^m = x^{(n-1)} \circ (y^m)$$

(where $x^{(0)} \circ y = y$), and by obvious induction we find that

$$(**) \quad 1 = x \circ (y^{m^{n-1}}) = (x \circ y)^{m^{n-1}} \text{ for all } y \text{ in } H'.$$

So $x \circ H'$ and H_3 are p -groups. Also $H' = H_3$, because the central quotient group of any group can not be locally cyclic. So $H' = G'/G''$ is a p -group, and by nilpotency of G' we deduce that G' and G are p -groups, showing (c). To show (d) we consider an element x of H which does not centralize H' . As $H' \not\subseteq Z(H)$ such elements exist. By equation (**) we find that

$$(H')^{m^{n-1}} \subseteq C(\{x\}),$$

and

$$H' \not\subseteq C(\{x\})$$

because x does not commute with all elements in H' . Now m is a power of p , and from the last two statements we conclude that

$$H' \neq (H')^{m^{n-1}}$$

and consequently

$$H' \neq (H')^p,$$

and (d) follows immediately.

COROLLARY 1. *If the torsionfree soluble group G satisfies condition (II), then G is nilpotent.*

This follows from (c) of Lemma 1.

LEMMA 2. *Assume that G is a non-abelian group possessing an abelian normal subgroup N of finite exponent p^k such that $G/N \cong C_{p^\infty}$. Then*

- (a) G is not hypercentral.
- (b) G satisfies condition (II) if and only if there is no subgroup $U \neq G$ such that $UN = G$.

Proof. The exponent of $G/C(Z_2(G) \cap N)$ does not exceed the exponent of N , and because $N \subseteq C(Z_2(G) \cap N)$ we have that $G/C(Z_2(G) \cap N)$ is a quotient group of G/N which has no proper quotient group of finite exponent. So $Z_2(G) \cap N \subseteq Z(G)$. If G is hypercentral, we would consequently have $N \subseteq Z(G)$ and G would be abelian, which is not true. So (a) is true.

If there is no subgroup $U \neq G$ such that $UN = G$, then for each proper subgroup V of G we have that VN is the extension of an abelian group of finite exponent p^k by a cyclic group of order p^r , so VN is nilpotent and V is a

subnormal nilpotent subgroup of VN . The subnormality of V in G now follows from the normality of VN in G , and we have shown (b) in one direction. The other direction follows from Lemma 1,a and Lemma 2,a.

COROLLARY 2. *If the hypercentral group G satisfies condition (II), then G is nilpotent.*

This follows from statements (b), (d) of Lemma 1 together with statement (a) of Lemma 2.

If G is the extension of an elementary abelian normal p -subgroup N generated by one class of conjugate elements such that $G/N \cong C_{p^\infty}$, then the endomorphism ring induced by G in N is some quotient ring of the group algebra of C_{p^∞} over Z_p , the field of p elements; and every normal subgroup of G in N corresponds to an ideal of this ring. The ideal structure of this algebra $Z_p[C_{p^\infty}]$ will therefore be of great interest to us.

LEMMA 3. *The set of all ideals of $Z_p[C_{p^\infty}]$ is completely ordered.*

Proof. It is sufficient to show the following statement: If u and v are two elements of $Z_p[C_{p^\infty}]$ and u is not in the ideal generated by v , then v belongs to the ideal generated by u . Now both u and v are contained in the subring generated by the elements of order p^k for some integer k , and in this subring the ideals are ordered; if $R = Z_p[\{x\}]$ is the ring considered, the ideals are just

$$(1-x)^s R \quad \text{for } 0 \leq s \leq p^k, \quad \text{because } 0 = 1 - x^{p^k} = (1-x)^{p^k}.$$

Therefore v is contained in the ideal of this subring which is generated by u , and the same applies in $Z_p[C_{p^\infty}]$.

We are now ready to construct our group and to prove its properties. For convenience we begin with a bigger group F containing the desired group G :

$$F = \left\{ a, x_i \mid \begin{array}{l} a^p = 1, \quad a^{(2)} \circ b = 1 \text{ for all } b \in F, \\ x_1^p = (x_1 a)^p = 1, \quad x_{i+1}^p = x_i a \text{ for all positive } i. \end{array} \right\}$$

This group F has an elementary abelian normal subgroup A which is generated by one conjugacy class, namely by the conjugates of a ; and $F/A \cong C_{p^\infty}$. F would be abelian, if the relation $a \circ b = 1$ were substituted for $a^{(2)} \circ b = 1$. This new group is a quotient group of F and is the direct product of a group isomorphic to C_{p^∞} with a group of order p . So $F/F' \not\cong F/A$ and $F' \neq A$. This disqualifies F as a candidate satisfying (I), (II) and (III). We now consider the subgroup

$$G = \{x_1 a, x_2 a, x_3 a, \dots\}$$

and show that G satisfies condition (II). That G satisfies conditions (I) and (III) also will be seen after the introduction of the second presentation of G . First we consider $(x_i a) \circ (x_{i+1} a) \in G'$.

By the relations of F we know that

$$\begin{aligned}(x_i a) \circ (x_{i+1} a) &= (x_{i+1}^p) \circ (x_{i+1} a) = (x_{i+1}^p) \circ a \\ &= (x_i a) \circ a = x_i \circ a.\end{aligned}$$

So $x_i \circ a$ is contained in G' for all i . Also G' is a normal subgroup of $F = GA$, because it is normal both in G and in A . But the smallest normal subgroup of F containing all the commutators $x_i \circ a$ is F' . Hence $F' = G'$.

The endomorphism ring induced by F in A is a quotient ring of the group algebra of C_{p^∞} over the field of p elements, and the same holds for the endomorphism ring induced by G in G' . Let us now denote $x_i a$ by z_i . Then the defining relations of G are

$$\begin{aligned}z_{i+1}^p &= (x_{i+1} a)^p = x_{i+1}^p (x_{i+1}^{(p-1)} \circ a) = z_i (z_{i+1}^{(p-1)} \circ a), \\ z_1^p &= 1, \quad (z_1 a^{-1})^p = z_1^p (z_1^{(p-1)} \circ a)^{-1} = 1.\end{aligned}$$

Every subgroup H of G with $HG' = G$ contains a subgroup H^* which can be expressed in the following form:

$$H^* = \{z_1 a^{P_1}, z_2 a^{P_2}, \dots\},$$

where a^Q denotes the image of a under the endomorphism Q . The elements a^{P_i} are contained in $A \cap G$, which is the smallest normal subgroup of G containing G' and all products $z_i^{-1} z_{i+1}^p = z_{i+1}^{(p-1)} \circ a$. From Lemma 3 we deduce $A \cap G = G' = F'$, and the elements a^{P_i} are contained in G' , they are therefore products of conjugates of elements $x_j \circ a$, and P_i is consequently a non-unit in the endomorphism ring. Suppose now that P_i can be expressed by powers of \tilde{z}_{i+k} , where \tilde{z}_{i+k} denotes the automorphism induced in A by conjugation with z_{i+k} . We evaluate

$$\begin{aligned}(z_{i+k+1} a^{P_{i+k+1}})^{p^{k+1}} &= (z_{i+k+1})^{p^{k+1}} a^{P_{i+k+1}(\tilde{z}_{i+k+1}-1)p^{k+1}-1} \\ &= (z_{i+k})^{p^k} a^{(1+P_{i+k+1})(\tilde{z}_{i+k+1}-1)p^{k+1}-1} \\ &= (z_{i+k-1})^{p^{k-1}} a^{(\tilde{z}_{i+k-1})^{p^k-1} + (1+P_{i+k+1})(\tilde{z}_{i+k+1}-1)p^{k+1}-1}\end{aligned}$$

and finally

$$(z_{i+k+1} a^{P_{i+k+1}})^{p^{k+1}} = z_i a^{T(i,k)},$$

where

$$T(i, k) = (\tilde{z}_{i+k+1} - 1)^{p^{k+1}-1} (1 + P_{i+k+1}) + \sum_{s=1}^k (\tilde{z}_{i+s} - 1)^{p^s-1}.$$

The element

$$(z_i a^{P_i})^{-1} (z_{i+k+1} a^{P_{i+k+1}})^{p^{k+1}} = a^{T(i,k)-P_i}$$

is contained in $H^* \cap G'$. As $H^* \cap G'$ is a normal subgroup of H^* and of G' , we have that $H^* \cap G'$ is normal in $H^* G' = G$. We want to find the smallest normal subgroup of G containing all $a^{T(i,k)-P_i}$. This is equivalent to the problem of finding the smallest ideal of the endomorphism ring induced by F in A containing all $T(i, k) - P_i$.

As P_{i+k+1} is not a unit, $1 + P_{i+k+1}$ is a unit by Lemma 3, and the elements

$$(\tilde{z}_{i+k+1} - 1)^{p^{k+1}-1} \quad \text{and} \quad (\tilde{z}_{i+k+1} - 1)^{p^{k+1}-1} (1 + P_{i+k+1})$$

generate the same ideal. On the other hand,

$$\sum_{s=1}^k (\tilde{z}_{i+s} - 1)^{p^s-1} - P_i$$

generates the same ideal as some $(\tilde{z}_{i+k} - 1)^m = (\tilde{z}_{i+k+1} - 1)^{pm}$, because it can be written as a polynomial in \tilde{z}_{i+k} . The ideals generated by

$$(\tilde{z}_{i+k+1} - 1)^{p^{k+1}-1} \quad \text{and by} \quad (\tilde{z}_{i+k+1} - 1)^{pm}$$

are different whenever at least one of them is different from the zero ideal. These two ideals are generated by the two summands of $T(i, k) - P_i$, and, by Lemma 3, the ideal generated by $T(i, k) - P_i$ contains both these elements. Hence it contains in particular

$$(\tilde{z}_{i+k+1} - 1)(\tilde{z}_{i+k+1} - 1)^{p^{k+1}-1} = \tilde{z}_i - 1.$$

This shows that H^* contains $z_i \circ a$ for all i , and $G' \subset H^*$. Now $H^* G' = G$ yields $H^* = G$. By construction of H^* we have that $H^* \subseteq H$, so $H = G$. By Lemma 2 this suffices to show that G satisfies condition (II) if G is non-abelian. If G were abelian, condition (II) would be satisfied trivially, but we will see soon that the centre of G is trivial.

For our second presentation of G we first consider the series of groups

$$R_i = \left\{ a_i, b_i \mid \begin{array}{l} a_i^p = 1, \quad a_i^{(2)} \circ y = 1 \text{ for all } y \in R_i, \\ b_i^{p^i} = 1, \quad b_i^{(p^i - p^{i-1} - 1)} \circ a_i = 1 \end{array} \right\}$$

The groups R_i possess subgroups

$$S_i = \{b_i^{(p-1)} \circ a_i, b_i^p(b_i^{(p-2)} \circ a_i)\}$$

(where $x^{(0)} \circ y = y$).

We find that

$$(b_i^p) \circ (b_i^{(k)} \circ a_i) = b_i^{(k+p)} \circ a_i$$

and

$$(b_i^p(b_i^{(p-2)} \circ a_i))^{p^{t-1}} = b_i^{p^{t+1}}(b_i^{(p(p^{t-1})+p-2)} \circ a_i).$$

We find the following relations in S_i :

$$(b_i^{(p-1)} \circ a_i)^p = 1, \quad (b_i^{(p-1)} \circ a_i)^{(2)} \circ y = 1 \text{ for all } y \in S_i,$$

$$(b_i^p(b_i^{(p-2)} \circ a_i))^{p^{i-1}} = b_i^{(p^{i-2})} \circ a_i = 1$$

$$\begin{aligned} (b_i^p(b_i^{(p-2)} \circ a_i))^{(p^{i-1}-p^{i-2}-1)} \circ (b_i^{(p-1)} \circ a_i) &= (b_i^p)^{(p^{i-1}-p^{i-2}-1)} \circ (b_i^{(p-1)} \circ a_i) \\ &= b_i^{(p^i-p^{i-1}-1)} \circ a_i = 1, \end{aligned}$$

$$1 \neq (b_i^p(b_i^{(p-2)} \circ a_i))^{p^{i-2}}$$

and

$$1 \neq (b_i^p(b_i^{(p-2)} \circ a_i))^{(p^{i-1}-p^{i-2}-2)} \circ (b_i^{(p-1)} \circ a_i).$$

So S_i is isomorphic to R_{i-1} , and there is an isomorphism of R_{i-1} onto S_i mapping a_{i-1} onto $b_i^{(p-1)} \circ a_i$ and b_i onto $b_i^p(b_i^{(p-2)} \circ a_i)$. Using these isomorphisms we embed successively R_{i-1} into R_i for all i . The then defined set theoretical union $\bigcup_{i=0}^{\infty} R_i$ satisfies condition (I) because all R_i do. To show that it satisfies condition (III), it is sufficient to show that

$$Z(R_i) \cap Z(R_{i+1}) = 1 \text{ for } p \neq 2$$

and that

$$Z(R_i) \cap Z(R_{i+2}) = 1 \text{ for } p = 2.$$

For $p \neq 2$ we find that

$$Z(R_i) = \{b_i^{(p^i-p^{i-1}-2)} \circ a_i\}.$$

By the embedding isomorphism we have that

$$\begin{aligned} b_i^{(p^i - p^{i-1} - 2)} \circ a_i &= (b_{i+1}^p (b_{i+1}^{(p-2)} \circ a_{i+1}))^{(p^i - p^{i-1} - 2)} \circ a_i \\ &= (b_{i+1}^p)^{(p^i - p^{i-1} - 2)} \circ (b_{i+1}^{(p-1)} \circ a_{i+1}) \\ &= b_{i+1}^{(p^{i+1} - p^i - p - 1)} \circ a_{i+1}, \end{aligned}$$

and this element is not contained in $Z(R_{i+1})$.

So $Z(R_i) \cap Z(R_{i+1}) = 1$ for $p \neq 2$.

For $p = 2$ and $i \neq 1$ we find that

$$Z(R_i) = \{b_i^{(2^{i-1} - 2)} \circ a_i, b_i^{2^{i-1}}\},$$

and by the embedding endomorphism we have that

$$b_i^{(2^{i-1} - 2)} \circ a_i = b_{i+1}^{(2^i - 3)} \circ a_{i+1}$$

and

$$b_i^{2^{i-1}} = (b_{i+1}^2 a_{i+1})^{2^{i-1}} = b_{i+1}^{2^i} (b_{i+1}^{(2^i - 2)} \circ a_{i+1}).$$

It follows immediately that

$$Z(R_i) \cap Z(R_{i+1}) = \{b_{i+1}^{2^i} (b_{i+1}^{(2^i - 2)} \circ a_{i+1})\}.$$

Finally

$$b_{i+1}^{2^i} (b_{i+1}^{(2^i - 2)} \circ a_{i+1}) = b_{i+2}^{2^{i+1}} (b_{i+2}^{(2^{i+1} - 2)} \circ a_{i+2}) (b_{i+2}^{(2^{i+1} - 3)} \circ a_{i+2}),$$

and this shows that $Z(R_i) \cap Z(R_{i+2}) = Z(R_i) \cap Z(R_{i+1}) \cap Z(R_{i+2}) = 1$.

The proof of the validity of condition (III) in $\bigcup_{i=1}^{\infty} R_i$ is now complete.

Keeping in mind that the relations

$$z_{i+1}^p = z_i(z_{i+1}^{(p-1)} \circ a)$$

and

$$z_{i+1}^p (z_{i+1}^{(p-2)} \circ (z_{i+1} \circ a)^{-1}) = z_i$$

are equivalent, the reader is now able to check without difficulty that the relations in $\bigcup_{i=1}^{\infty} R_i$ for a_i, b_i are exactly the same as for $(z_i \circ a)^{-1}, z_i$ in G , so $\bigcup_{i=1}^{\infty} R_i$ and G are isomorphic and $\bigcup_{i=1}^{\infty} R_i \cong G$ satisfies conditions (I), (II) and (III).

We summarize our result in the

THEOREM. *For each prime p there is a metabelian p -group G with trivial centre, such that every proper subgroup of G is nilpotent and subnormal.*

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